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ON A WELL-POSED TURBULENCE MODEL

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Abstract. This report considers mathematical properties, important for practical computations, of a model for the simulation of the motion of large eddies in a turbulent flow. In this model, closure is accomplished in the very simple way:

$$\overline{\mathbf{u}\mathbf{u}} \sim \overline{\bar{\mathbf{u}}\bar{\mathbf{u}}}, \text{ yielding the model}$$
$$\nabla \cdot \mathbf{w} = 0, \quad \mathbf{w}_t + \nabla \cdot (\overline{\mathbf{w}\mathbf{w}}) - \nu \Delta \mathbf{w} + \nabla q = \bar{\mathbf{f}}.$$

In particular, we prove existence and uniqueness of strong solutions, develop the regularity of solutions of the model and give a rigorous bound on the modelling error, $||\bar{\mathbf{u}} - \mathbf{w}||$. Finally, we consider the question of non-physical vortices (false eddies), proving that the model correctly predicts that only a small amount of vorticity results when the total turning forces on the flow are small.

1. **Introduction.** The great challenge in simulation of turbulent flows from applications ranging from geophysics to biomedical device design is that equations for the pointwise flow quantities are well-known but intractable to computational solution and sensitive to uncertainties and perturbation in problem data. On the other hand, closed equations for the averages of flow quantities cannot be obtained directly from the physics of fluid motion. Thus, modeling in large eddy simulation (meaning the approximation of local, spacial averages in a turbulent flow) is typically based on guesswork (phenomenology), calibration (data fitting model parameters) and (at best) approximation.

If $\mathbf{u}(t,x) = (u^1(t,x), u^2(t,x), u^3(t,x))$ and $p(t,x)$ are the velocity and pressure in an incompressible turbulent flow, then $(\mathbf{u}, p)$ satisfy the Navier-Stokes equation for $t > 0$,

$$\mathbf{u}_t + \nabla \cdot (\mathbf{u}\mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \mathbb{R}^3 \tag{1}$$

where $\mathbf{f}$ is the body force driving the flow, $\nabla\cdot$ is the divergence operator, $\mathbf{u}\mathbf{u}$ is the tensor $(u^i u^j)_{1 \le i,j \le 3}$. If overbar denotes a local, spacial averaging operator that

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commutes with differentiation, then, averaging (1) gives the following non-closed equations for $\overline{\mathbf{u}}, \overline{p}$ in $]0,\infty[\times I\!R^3$:

$$\nabla \cdot \overline{\mathbf{u}} = 0 \quad \text{and} \quad \overline{\mathbf{u}}_t + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} = \overline{\mathbf{f}}. \tag{2}$$

The famous closure problem which we study herein arises because $\overline{\mathbf{u}\,\mathbf{u}} \neq \mathbf{u}\,\mathbf{u}$. To isolate the turbulence closure problem from the difficult question of wall laws for near wall turbulence, we study (2) subject to periodic boundary conditions (and zero mean)

$$\left\{ \begin{array}{l} \mathbf{u}(x + L\mathbf{e}_j, t) = \mathbf{u}(t, x) \\ \displaystyle\int_Q \mathbf{u}_0(x)dx = \int_Q \mathbf{u}(t,x)dx = \int_Q \mathbf{f}(t,x)\ dx = 0, \end{array} \right. \tag{3}$$

where $Q = [0, L]^3$. The closure problem is to replace the tensor $\overline{\mathbf{u}\,\mathbf{u}}$ with a tensor $S(\mathbf{u}, \mathbf{u})$ depending only on $\overline{\mathbf{u}}$ (and not $\mathbf{u}$). There are very many closure models proposed in large eddy simulation (or LES) (see Sagaut [21] and [10] for examples) reflecting the centrality of closure in turbulence simulation. Calling $\mathbf{w}, q$ the resulting approximations to $\overline{\mathbf{u}}, \overline{p}$, we are led to considering the model,

$$\nabla \cdot \mathbf{w} = 0 \quad \text{and} \quad \mathbf{w}_t + \nabla \cdot S(\mathbf{w}, \mathbf{w}) - \nu\Delta\mathbf{w} + \nabla q = \overline{\mathbf{f}}. \tag{4}$$

With any reasonable averaging operator, the true averages, $\overline{\mathbf{u}}, \overline{p}$ are smoother than $\mathbf{u}, p$. Thus, solutions of any derived model such as (4) <u>should</u> be more regular than the Navier-Stokes equations. However, in spite of the intense interest in closure models for turbulence, there are very few whose mathematical development even parallels that of the NSE, e.g., [18], [10], [11] [21], [16].

In this report, we consider the simplest, accurate closure model. If $\mathbf{u}$ is a constant flow then $\mathbf{u} = \overline{\mathbf{u}}$. The simple closure model (that is exact on constant flows) is

$$\overline{\mathbf{u}\mathbf{u}} \cong \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}}\ (=:\ S(\overline{\mathbf{u}},\ \overline{\mathbf{u}})), \tag{5}$$

leading to

$$\nabla \cdot \mathbf{w} = 0 \quad \text{and} \quad \mathbf{w}_t + \nabla \cdot (\overline{\mathbf{w}\mathbf{w}}) - \nu\Delta\mathbf{w} + \nabla q = \overline{\mathbf{f}}. \tag{6}$$

In some sense, (5) is the most basic (hence zeroth) model in LES. It can arise by dropping the cross and Reynolds terms and keeping only the Leonard/resolved term [12]. It is the zeroth Stolz-Adams ADM model [19], [20]. It is the rational model [7] truncated to $O(\delta^2)$ terms.

We shall show that the model (6) has the mathematical properties which are expected of a model derived from the NSE by an averaging operation and which are important for practical computations using (6).

The averaging operator chosen in (6) is a differential filter, [6], [3], [7], [16]. Let $\delta > 0$ denote the averaging radius (typically related to the finest computationally feasible mesh used in a simulation of (6)). Given a periodic function $\phi(x) \in L^2(\Omega)$, define its average $\overline{\phi}$ to be the unique Q-periodic solution of

$$A\overline{\phi} := -\delta^2\Delta\overline{\phi} + \overline{\phi} = \phi \quad \text{in } I\!R^3, \quad \phi(x + L\mathbf{e}_j,) = \phi(x).$$

Because of the zeroth order term in A, this periodic problem is well posed without the assumption of zero mean on the RHS. With this averaging, the model (5) has consistency $O(\delta^2)$:

$$\overline{\mathbf{u}\mathbf{u}} = \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} + O(\delta^2), \text{ for smooth } \mathbf{u}.$$

We prove that the model (6) has a <u>unique, strong solution</u> and that the smoothness of this solution is limited only by the smoothness of the problem data $\mathbf{u}_0$ and $\mathbf{f}$.

These properties are essential for numerical simulations. We prove that as $\delta \to 0$, the solution of the model is such that $\mathbf{w} \to \mathbf{u}$ (the solution of the NSE) in the appropriate sense. This property is critical for consistency of the solution of the model with the true flow averages. Finally, we introduce the question of spurious vorticity/eddies generated by the model and give one weakly positive result.

The development of these mathematical properties is based upon a skew symmetry property of the model's nonlinearity and the energy estimate it induces. To be specific, for sufficiently smooth functions which are periodic and divergence free

$$\int_Q \nabla \cdot (\overline{\mathbf{w}\mathbf{w}}) \cdot A\mathbf{w}\, dx = \int_Q \nabla \cdot (\mathbf{w}\mathbf{w}) \cdot A^{-1}A\mathbf{w}\, dx = \int_Q \nabla \cdot (\mathbf{w}\mathbf{w}).\mathbf{w}\, dx = 0.$$

Thus, (loosely speaking) multiplying the model by $A\mathbf{w}$ and integrating over the domain shows that $\mathbf{w}$ satisfies the very strong stability property

$$\begin{aligned} &\sup_{0<t<T} \{||\mathbf{w}(t,\cdot)||^2 + \delta^2 ||\nabla \mathbf{w}(t,\cdot)||^2\} \\ &+\nu \int_0^T (||\nabla \mathbf{w}(t,\cdot)||^2 + \delta^2 ||\Delta \mathbf{w}(t,\cdot)||^2) dt \leq C(\nu, T, \overline{\mathbf{u_0}}, \mathbf{f}), \end{aligned} \tag{7}$$

where $||\cdot||$ is the L^2 norm defined by

$$\forall\, \mathbf{v} \in (L^2_{loc}(\mathbb{R}^3))^3, \quad \mathbf{v} \text{ is } Q-\text{periodic}, \quad ||\mathbf{v}||^2 = \int_Q |\mathbf{v}(x)|^2 dx.$$

This property is also shared by suitably defined weak solutions of the model proven to exist in [17]. Exploiting this strong stability property, we shall first prove existence and regularity of *strong* solutions to the model. Before giving the statement of our main result, we first define the spaces we shall use. Being given $k \geq 0$, let W_k denote the space

$$W_k = \left\{ \mathbf{v} \in (H^k_{loc}(\mathbb{R}^3))^3,\, \mathbf{v} \text{ is } Q-\text{periodic}, \int_Q \mathbf{v} = 0 \right\}, \tag{8}$$

and let V_k denote the space

$$V_k = \{\mathbf{v} \in W_k,\, \nabla \cdot \mathbf{v} = 0\}. \tag{9}$$

We shall also consider the spaces

$$\mathcal{W} = \left\{ \mathbf{v} \in (C^\infty_{loc}(\mathbb{R}^3))^3,\, \mathbf{v} \text{ is } Q-\text{periodic}, \int_Q \mathbf{v} = 0 \right\}, \tag{10}$$

and

$$\mathcal{V} = \{\mathbf{v} \in \mathcal{W},\, \nabla \cdot \mathbf{v} = 0\}. \tag{11}$$

Theorem 1. *For any* $\mathbf{u}_0 \in V_0$, $\mathbf{f} \in L^2([0,T], V_1')$ *the model* (6) *has a unique solution*

$$(\mathbf{w}, q) \in [L^2([0,T], V_2) \cap L^\infty([0,T], V_1] \times L^2([0,T], L^2_{loc}(\mathbb{R}^3)), \tag{12}$$

where for all $t \in]0,T]$, $\int_Q q(t,x) dx = 0$ *and the energy equality holds:*

$$\left\{ \begin{aligned} &\frac{1}{2}\int_Q (|\mathbf{w}(t,x)|^2 + \delta^2 |\nabla \mathbf{w}(t,x)|^2) dx \\ &+\nu \int_0^t \int_Q (|\nabla \mathbf{w}(x,t')|^2 + \delta^2 |\Delta \mathbf{w}(x,t')|^2) dx dt' \\ &= \frac{1}{2}\left(\int_Q (|\overline{\mathbf{u}}_0(x)|^2 + \delta^2 |\nabla \overline{\mathbf{u}}_0(x)|^2) dx \right) + \int_0^t \int_Q \mathbf{f}.\mathbf{w}\, dx dt'. \end{aligned} \right. \tag{13}$$

In addition, let $k \geq 0$. If $\mathbf{u}_0 \in V_{\sup(0,k-1)}$ and $\mathbf{f} \in L^2([0,T], W_{k-1})$, then the solution is such that $\mathbf{w} \in L^2([0,T], V_{k+2}) \cap L^\infty([0,T], V_{k+1})$ and $q \in L^2([0,T], H^k_{loc}(\mathbb{R}^3))$.

Remark 1. On the left-hand side of (13) for δ fixed and the viscosity $\nu \to 0$, we retain a quite strong regularity property $\mathbf{w} \in L^\infty(0,T;V_1)$. Using this observation, existence can also be proven for the Euler model that arises by setting the viscosity coefficient $\nu = 0$ in (6). (This fact was pointed out to the author W.L. by D. D. Holm and E. Titi.)

Corollary 1. *Consider the model (6) with $\nu = 0$ and $\delta > 0$. For any $\mathbf{u}_0 \in V_0$, $\mathbf{f} \in L^2([0,T], W_0)$, the Euler LES model*

$$\nabla \cdot \mathbf{w} = 0, \quad \textit{and} \quad \mathbf{w}_t + \nabla \cdot (\overline{\mathbf{w}\mathbf{w}}) + \nabla q = \overline{\mathbf{f}}$$

has a unique weak solution. That weak solution satisfies the energy equality:

$$\begin{aligned} &\frac{1}{2}(||\mathbf{w}(t,\cdot)||^2 + \delta^2||\nabla \mathbf{w}(t,\cdot)||^2) \qquad (14) \\ = \; &\frac{1}{2}\left(\int_Q (|\overline{\mathbf{u}}_0(x)|^2 + \delta^2|\nabla\overline{\mathbf{u}}_0(x)|^2)dx\right) + \int_0^t \int_Q \mathbf{f}(t',x)\,\mathbf{w}(t',x)\,dxdt'. \end{aligned}$$

One of the most important criteria in evaluating a model is that it be accurate. Yet, there are few analytical studies of $||\mathbf{w} - \overline{\mathbf{u}}||$ primarily because of deficiencies in the analytical tools available. We are not able herein to give a complete and comprehensive, á priori proof of the model's accuracy. Nevertheless, we prove some partial results that confirm that the model has properties expected of one derived by averaging from the NSE. For example, we show in Section 3 that as $\delta \to 0$ there is a subsequence δ_j with

$$\mathbf{w} \to \mathbf{u}, \text{ a weak solution of the NSE, as } \delta_j \to 0$$

and if that weak solution $\mathbf{u}$ is unique

$$\mathbf{w} \to \mathbf{u} \quad \text{as} \quad \delta \to 0.$$

This result addresses the issue of "consistency in the limit" [11] as $\delta \to 0$ of the model. The model (6) and Camassa-Holm model ([5]) have a similar energy balance and both satisfy a limit-consistency result.

Let $\overline{\tau}$ denote the modeling consistency error tensor

$$\overline{\tau}(\mathbf{u},\mathbf{u}) := \overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} - \overline{\mathbf{u}\mathbf{u}}$$

then it is straightforward to see that the true flow averages $\overline{\mathbf{u}}$ satisfy

$$\overline{\mathbf{u}}_t + \nabla \cdot (\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}}) - \nu\Delta\overline{\mathbf{u}} + \nabla\overline{p} = \overline{\mathbf{f}} + \nabla \cdot \overline{\tau}$$

and the error in the model $\mathbf{e} = \overline{\mathbf{u}} - \mathbf{w}$ satisfies an equation driven only by the averaged consistency error $\nabla \cdot \overline{\tau}$:

$$\mathbf{e}_t + \nabla \cdot (\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{w}\mathbf{w}}) - \nu\Delta\mathbf{e} + \nabla(\overline{p} - q) = \nabla \cdot \overline{\tau}.$$

Thus, $||\mathbf{e}||$ being small depends upon two factors: a small consistency error, $||\overline{\tau}||$ small, and a strong enough stability property that $||\mathbf{e}||$ is bounded by some norm of $\overline{\tau}$. If the stability constants in this bound are to be independent of δ, then (with the analytic tools available at this time) an extra condition ensuring global uniqueness of $\mathbf{u}$ is necessary. In Section 3, we prove such a bound (which ensures the model is verifiable in the sense of [10]. The other criteria is that $||\overline{\tau}||$ is small as $\delta \to 0$. This is often performed by computational experiments, see the discussion in [9]. Herein, we give in Section 3, analytical bounds verifying that $\overline{\tau} \to 0$ (with rates) as

$\delta \to 0$ for smooth enough solutions $\mathbf{u}$ of the NSE. One main open question is that the natural norm on $\overline{\tau}$ for verifiability is stronger than the natural norm on $\overline{\tau}$ for evaluating the model's consistency error.

Lastly, we consider the question of spurious vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{w}$. Our result on this question is positive but weak. We show that if $\nabla \times \mathbf{u}_0 \equiv 0$ and $\nabla \times \mathbf{f} \equiv 0$ then (correctly) $\boldsymbol{\omega} \equiv 0$ and that the zero vorticity state is stable: if $\nabla \times \mathbf{u}_0 \equiv 0$ and $\nabla \times \mathbf{f} \equiv 0$ are both small then $\boldsymbol{\omega}$ is comparably small.

2. Uniqueness, Regularity and Stability of the Model.

2.1. **Background.** Let $p \geq 1$, and let $L^p_\sharp$ the space defined by

$$L^p_\sharp = \left\{\varphi \in L^p_{loc}(\mathbb{R}^3),\ \varphi \text{ is } Q - \text{periodic}\right\}$$

equipped with the norm

$$||\varphi||_{L^p_\sharp} = \left(\int_Q |\varphi(x)|^p dx\right)^{\frac{1}{p}}.$$

Let $W^{k,p}_\sharp$ $(k \geq 1)$ denote the space defined by

$$W^{k,p}_\sharp = \left\{\varphi \in W^{k,p}_{loc}(\mathbb{R}^3),\ \varphi \text{ is } Q - \text{periodic}\right\},$$

equipped with the norm

$$||\varphi||_{W^{k,p}_\sharp} = \sum_{q=1}^{k} \left(\int_Q |\nabla^q \varphi(x)|^p dx\right)^{\frac{1}{p}} + ||\varphi||_{L^p_\sharp}.$$

We also note $H^k_\sharp = W^{k,2}_\sharp$. Notice that if

$$W^{k,p}_{\sharp,0} = \left\{\varphi \in W^{k,p}_\sharp,\ \int_Q \varphi(x) dx = 0\right\},$$

then the space $W^{k,p}_{\sharp,0}$ can be equipped with the norm

$$||\varphi||_{W^{1,p}_{\sharp,0}} = \sum_{q=1}^{k} \left(\int_Q |\nabla^q \varphi(x)|^p dx\right)^{\frac{1}{p}}.$$

In particular, the space W_k introduced by the definition (8) is the space $(W^{2,k}_{\sharp,0})^3 = (H^k_{\sharp,0})^3$.

The averaging operator A is defined by

$$A\overline{\phi} := -\delta^2 \Delta \overline{\phi} + \overline{\phi} = \phi \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad \phi(x + L\mathbf{e}_j) = \phi(x). \tag{15}$$

defining an operator $A : W^{1,p}_\sharp \to (W^{1,p'}_\sharp)'$. One easily sees that A is self-adjoint and has the regularity property

$$\forall\, r, \quad \forall\, \phi \in H^r_\sharp, \quad \overline{\phi} = A^{-1}\phi \in H^{r+2}_\sharp. \tag{16}$$

Recall that the function spaces W_k, V_k, $\mathcal{W}$ and $\mathcal{V}$ are defined by (8), (9), (10) and (11). In particular $V_k = \{\mathbf{v} \in W_k, \nabla \cdot \mathbf{v} = 0\}$. It has been shown in [17] that when

$$\mathbf{u}_0 \in V_0, \quad \mathbf{f} \in L^2([0,T], V_1') \tag{17}$$

then (6) has a weak solution $(\mathbf{w}, q)$ where $\mathbf{w}$ is such that

$$\mathbf{w} \in L^2([0,T], V_2) \cap L^\infty([0,T], V_1).$$

Notice such a result makes sense because light modification of the results in [8] yield $\overline{\mathcal{V}} = V_1$.

Throughout the section, we assume that (17) holds. The pressure term is subjected to satisfy $\displaystyle\int_Q q = 0$.

2.2. **Uniqueness.** We first prove the following uniqueness result.

Theorem 2. *Assume that* (17) *holds. Then there exists at most one solution to* (6).

Proof. Let $(\mathbf{w}_1, q_1)$ and $(\mathbf{w}_2, q_2)$ be two solutions to (6). Write $\boldsymbol{\phi} = \mathbf{w}_2 - \mathbf{w}_1$, $r = q_2 - q_1$. Thus $\boldsymbol{\phi}$ is solution to the problem

$$\left\{\begin{array}{l} \boldsymbol{\phi}_t + \nabla \cdot (\overline{\mathbf{w}_2\mathbf{w}_2 - \mathbf{w}_1\mathbf{w}_1}) - \nu \triangle \boldsymbol{\phi} + \nabla r = 0, \\ \nabla \cdot \boldsymbol{\phi} = 0, \\ \boldsymbol{\phi}_{t=0} = 0, \end{array}\right. \tag{18}$$

subject to periodic boundary conditions with zero mean. Notice that by using Schartz rule in the absence of boundaries one has in the sense of the distributions (see in [22]),

$$\nabla \cdot (\overline{\mathbf{w}_2\mathbf{w}_2 - \mathbf{w}_1\mathbf{w}_1}) = A^{-1} \nabla \cdot (\mathbf{w}_2\mathbf{w}_2 - \mathbf{w}_1\mathbf{w}_1).$$

Using $A\boldsymbol{\phi}$ as test function in (18) and integrating in space on a cell yields

$$\left\{\begin{array}{l} \displaystyle\frac{d}{2dt}\int_Q (|\boldsymbol{\phi}|^2 + \delta^2 |\nabla \boldsymbol{\phi}|^2) + \nu \int_Q (|\nabla \boldsymbol{\phi}|^2 + \delta^2 |\Delta \boldsymbol{\phi}|)^2 \\ \displaystyle = -\int_Q A^{-1}\nabla \cdot (\mathbf{w}_2\mathbf{w}_2 - \mathbf{w}_1\mathbf{w}_1).\, A\boldsymbol{\phi} \end{array}\right. \tag{19}$$

We focus our attention on the r.h.s of (19). By using self-adjointness of A one has

$$\int A^{-1}\nabla \cdot (\mathbf{w}_2\mathbf{w}_2 - \mathbf{w}_1\mathbf{w}_1).\, A\boldsymbol{\phi} = \int \nabla \cdot (\mathbf{w}_2\mathbf{w}_2 - \mathbf{w}_1\mathbf{w}_1).\, \boldsymbol{\phi}.$$

Furthermore, using the incompressibility constraint, one obtains after an easy algebraic computation and an integration by parts,

$$\int_Q \nabla \cdot (\mathbf{w}_2\mathbf{w}_2 - \mathbf{w}_1\mathbf{w}_1).\, \boldsymbol{\phi} = -\int_Q (\boldsymbol{\phi}\nabla)\boldsymbol{\phi}.\, \mathbf{w}_1.$$

Finally,

$$\frac{d}{2dt}\int_Q (|\boldsymbol{\phi}|^2 + \delta^2 |\nabla \boldsymbol{\phi}|^2) + \nu \int_Q (|\nabla \boldsymbol{\phi}|^2 + \delta^2 |\Delta \boldsymbol{\phi}|)^2 = \int_Q (\boldsymbol{\phi}\nabla)\boldsymbol{\phi}.\, \mathbf{w}_1. \tag{20}$$

By Cauchy-Schwarz inequality,

$$\left|\int_Q (\boldsymbol{\phi}\nabla)\boldsymbol{\phi} \cdot \mathbf{w}_1\right| \le ||\mathbf{w}_1||_{(L^4_\sharp)^3} ||\boldsymbol{\phi}||_{(L^4_\sharp)^3} ||\nabla\boldsymbol{\phi}||_{(L^2_\sharp)^3}.$$

Since

$$\mathbf{w}_1, \mathbf{w}_2, \boldsymbol{\phi} \in L^2([0,T], V_2) \cap L^\infty([0,T], V_1) \subset L^\infty([0,T], (L^4_\sharp)^3),$$

(by using Sobolev embedding theorem) it follows that

$$||\mathbf{w}_1||_{(L^4_\sharp)^3} ||\boldsymbol{\phi}||_{(L^4_\sharp)^3} ||\nabla\boldsymbol{\phi}||_{(L^2_\sharp)^3} \le C ||\nabla\boldsymbol{\phi}||^2_{(L^2_\sharp)^3},$$

where C is a constant which only depends on the data $\mathbf{f}$ and $\mathbf{u}_0$. Finally, with $C' = C'(\delta)$

$$C||\nabla\boldsymbol{\phi}||^2_{L^2} \le C'(\int_Q (|\boldsymbol{\phi}|^2 + \delta^2|\nabla\boldsymbol{\phi}|^2)).$$

By putting all of this together, one sees that (20) implies that

$$\frac{d}{2dt}\int_Q (|\boldsymbol{\phi}|^2 + \delta^2|\nabla\boldsymbol{\phi}|^2) \leq C'(\int_Q (|\boldsymbol{\phi}|^2 + \delta^2|\nabla\boldsymbol{\phi}|^2)).$$

Since $|\boldsymbol{\phi}|^2+\delta^2|\nabla\boldsymbol{\phi}|^2$ vanishes when $t=0$, Gronwall's Lemma implies that it vanishes for almost every t. Hence, uniqueness follows and the theorem is proven. □

2.3. **Regularity.** The aim of this subsection is the proof of the following regularity.

Theorem 3. *Let $k \geq 0$. Assume that $u_0 \in V_{\sup(0,k-1)}$ and $\mathbf{f} \in L^2([0,T], W_{k-1})$. Then the solution $(\mathbf{w}, q)$ of problem (6) is such that*

$$\begin{cases} \mathbf{w} \in L^2([0,T], V_{k+2}) \cap L^\infty([0,T], V_{k+1}), \\ q \in L^2([0,T], H^k_\sharp). \end{cases} \tag{21}$$

In the result above, one uses the convention $W_{-1} = V_1'$.

Corollary 2. *When $k \geq 2$, $\mathbf{w}$ is continuous in time and space. When $\mathbf{f}$ and $\mathbf{u}_0$ are C^∞, then $(\mathbf{w}, q)$ is C^∞ in space and time.*

Proof. In the following, D^k denotes any partial derivative of total order k. The result is already proven when $k=0$. Let $k \geq 1$. Assume

$$\mathbf{u}_0 \in V_{\sup(0,k-1)} \quad \text{and} \quad \mathbf{f} \in L^2([O,T], W_{k-1}). \tag{22}$$

Assume also that:

$$\begin{cases} \text{for any} \quad q = 0, ..., k-1 \\ \text{and for any derivative operator of order } q, D^q, \\ D^q\mathbf{w} \in L^2([0,T], V_2) \cap L^\infty([0,T], V_1). \end{cases} \tag{23}$$

In addition to (23), one assumes that all the constants involved only depend on the data $\mathbf{u}_0$ and $\mathbf{f}$.

It remains to prove that (23) implies

$$D^k\mathbf{w} \in L^2([0,T], V_2) \cap L^\infty([0,T], V_1). \tag{24}$$

By taking the k^{th} derivative of (6) and using the classical Schwartz rule, one has in the sense of the distributions

$$\begin{cases} D^k\mathbf{w}_t + \nabla\cdot(\overline{D^k(\mathbf{ww})}) - \nu\triangle D^k\mathbf{w} + \nabla D^k q = D^k\overline{\mathbf{f}}, \\ \nabla\cdot D^k\mathbf{w} = 0, \\ D^k\mathbf{w}_{t=0} = D^k\mathbf{w}_0 = D^k\overline{\mathbf{u}_0}, \end{cases} \tag{25}$$

where boundary conditions remain periodic and still with zero mean and the initial condition with zero divergence and mean. Taking $AD^k\mathbf{w}$ as test function in (25) and using the self-adjointness of the operator A yields

$$\begin{cases} \dfrac{d}{2dt}\displaystyle\int_Q (|D^k\mathbf{w}|^2 + \delta^2|\nabla D^k\mathbf{w}|^2) + \nu\int_Q (|\nabla D^k\mathbf{w}|^2 + \delta^2|\Delta D^k\mathbf{w}|)^2 \\ = \ _{V_1'}< D^k\mathbf{f}, D^k\mathbf{w} >_{V_1} - \displaystyle\int_Q \nabla\cdot(\overline{D^k(\mathbf{ww})}).\, AD^k\mathbf{w}, \end{cases} \tag{26}$$

where we note that because $\mathbf{f} \in L^2([0,T], W_{k-1})$, then $D^k\mathbf{f} \in L^2([0,T], V_1')$.

One first notes that

$$\begin{cases} |\ _{V_1'}< D^k\mathbf{f}, D^k\mathbf{w} >_{V_1}| \leq ||D^k\mathbf{f}||_{V_1'}||D^k\mathbf{w}||_{V_1} \\ \leq \dfrac{1}{2\nu}||D^k\mathbf{f}||_{V_1'} + \dfrac{\nu}{2}\displaystyle\int_Q |\nabla D^k\mathbf{w}|^2. \end{cases} \tag{27}$$

The second term of the r.h.s of (26) has to be estimated.

We show in the following that (23) implies

$$\left| \int_0^T \int_Q \nabla \cdot (\overline{D^k(\mathbf{ww})}).\, AD^k \mathbf{w} \right| \leq C, \tag{28}$$

where the constant C involves only the data $\mathbf{f}$ and $\mathbf{u}_0$. Inequality (28) combined with (26) and (27) gives obviously (24).

First, by Schwartz rule,

$$\nabla \cdot (\overline{D^k(\mathbf{ww})}).\, AD^k \mathbf{w} = A^{-1} \nabla \cdot (D^k(\mathbf{ww})).\, AD^k \mathbf{w}.$$

Then,

$$\int_Q \nabla \cdot (\overline{D^k(\mathbf{ww})}).AD^k \mathbf{w} = \int_Q \nabla \cdot (D^k(\mathbf{ww})).\, D^k \mathbf{w}. \tag{29}$$

As one notes $\mathbf{w} = (w_1, w_2, w_3)$,

$$\nabla \cdot (D^k(\mathbf{ww})).\, D^k \mathbf{w} = \partial_j (D^k(w_i w_j)) D^k w_i,$$

(summation convention). Leibnitz formula reads

$$D^k(w_i w_j) = C_k^q D^q w_i D^{k-q} w_j.$$

When combining this with the constraint $\nabla \cdot D^{k-q} \mathbf{w} = 0$, one has

$$\nabla \cdot (D^k(\mathbf{ww})).\, D^k \mathbf{w} = C_k^q (\partial_j D^q w_i) D^{k-q} w_j D^k w_i.$$

Therefore, k being fixed,

$$\int_Q \nabla \cdot (\overline{D^k(\mathbf{ww})}).AD^k \mathbf{w} = C_k^q \int_Q (\partial_j D^q w_i) D^{k-q} w_j D^k w_i. \tag{30}$$

In the summation with respect to the q index of the r.h.s of (30), one treats the case $q \geq 1$ and $q = 0$ one after each other.

Case $q \geq 1$. By the induction hypothesis (23),

$$D^{k-q} w_j \in L^\infty([0,T], H^1_\sharp) \subset L^\infty([0,T], L^6_\sharp)$$

by using Sobolev imbedding theorem. Furthermore, always by (23), $D^k w_i \in L^2([0,T], H^1) \cap L^\infty([0,T], L^2)$. Classical interpolation inequalities using Hölder inequality (see [14]) implies

$$D^k w_i \in L^2([0,T], H^1_\sharp) \cap L^\infty([0,T], L^2_\sharp) \subset L^4([0,T], L^3_\sharp).$$

Finally always by (23),

$$\partial_j D^q w_i \in L^2([0,T], L^2_\sharp).$$

Putting all together and using Hölder inequality shows that for fixed $q \geq 1$, i and j,

$$(\partial_j D^q w_i) D^{k-q} w_j D^k w_i \in L^2([0,T], L^1_\sharp) \subset L^1([0,T], L^1_\sharp).$$

Therefore,

$$\left| \int_0^T \int_Q (\partial_j D^q w_i) D^{k-q} w_j D^k w_i \right| \leq C, \tag{31}$$

for a constant C which only depends on the data $\mathbf{f}$ and $\mathbf{u}_0$.

Case $q = 0$. One has to consider the term $\partial_j w_i D^k w_j D^k w_i$ for fixed index. On one hand, one still has $\mathbf{w} \in L^2([0,T], V_2) \cap L^\infty([0,T], V_1)$. Therefore,

$$\partial_j w_i \in L^\infty([0,T], L^2_\sharp).$$

On the other hand, by (23),

$$D^k w_i, D^k w_j \in L^2([0,T], L^6_\sharp).$$

Therefore,

$$\partial_j w_i D^k w_j D^k w_i \in L^1([0,T], L^{6/5}_\sharp) \subset L^1([0,T], L^1_\sharp),$$

yielding

$$\left| \int_0^T \int_Q \partial_j w_i D^k w_j D^k w_i \right| \leq C, \tag{32}$$

for a constant C which only depends on the data $\mathbf{f}$ and $\mathbf{u}_0$.

When combining (31) and (32) to (30), one have proven (28) and the proof is finished.

The regularity of the pressure term is deduced from classical considerations, e.g., [1], [24].

The corollary is a classical consequence of Theorem 3. □

3. Accuracy of the Model.

3.1. **Orientation.** There are many questions that now arise. The first concerns the consistency error; we show herein that the solution of (6),

$$\nabla \cdot \mathbf{w} = 0, \quad \mathbf{w}_t + \nabla \cdot (\overline{\mathbf{w}\mathbf{w}}) - \nu \Delta \mathbf{w} + \nabla q = \overline{\mathbf{f}},$$

converges to a weak solution to the Navier-Stokes equations when δ goes to zero (stated precisely in Theorem 4 below) proving that the model is consistent in the limit as $\delta \to 0$.

Let $\boldsymbol{\tau}$ denote the model's consistency error

$$\boldsymbol{\tau}(\mathbf{u}, \mathbf{u}) := \overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{u}\mathbf{u}, \tag{33}$$

where $\mathbf{u}$ is a solution of the Navier-Stokes equations obtained as limit of a subsequence of the sequence $(\mathbf{w}_\delta)_{\delta>0}$.

We also prove in Theorem 5 that $||\overline{\mathbf{u}} - \mathbf{w}||_{L^\infty(0,T;V_0)}$ is bounded by $||\boldsymbol{\tau}||_{L^2([0,T],L^2_\sharp)}$.

We turn to estimates of $||\boldsymbol{\tau}||$ in the next section.

3.2. **Limit Consistency of the Model.** Throughout the section, we assume that (2.3) holds. Let $(\mathbf{w}_\delta, q_\delta)$ be the solution of (6) for a fixed δ.

Theorem 4. *There is a subsequence $\delta_j \to 0$ as $j \to \infty$ such that*

$$(\mathbf{w}_{\delta_j}, q_{\delta_j}) \to (\mathbf{u}, p) \quad as \quad \delta_j \to 0$$

where

$$(\mathbf{u}, p) \in [L^\infty([0,T], V_0) \cap L^2([0,T], V_1)] \times L^{\frac{4}{3}}([0,T], L^2_\sharp) \tag{34}$$

is a weak solution of the Navier-Stokes equations in $I\!R^3$ with periodic boundary conditions and zero mean value constraint. The sequence $(\mathbf{w}_{\delta_j})_{j\in I\!N}$ converges strongly to $\mathbf{u}$ in the space $L^{\frac{4}{3}}([0,T], (L^2_\sharp)^3)$ and weakly in $L^2([0,T], V_1)$ while the sequence $(q_{\delta_j})_{j\in I\!N}$ converges weakly to p in the space $L^{\frac{4}{3}}([0,T], L^2_\sharp)$.

Before proving Theorem 4, we first prove the following two lemmas.

Lemma 1. *Let* $(h_\delta)_{\delta>0}$ *be a sequence in* $L^{\frac{4}{3}}([0,T],L^2_\sharp)$. *Assume that* $(h_\delta)_{\delta>0}$ *converges weakly to some* h *in the space* $L^{\frac{4}{3}}([0,T],L^2_\sharp)$ *when* δ *goes to zero. Then the sequence* $(\overline{h}_\delta)_{\delta>0}$ *converges weakly to* h *in the space* $L^{\frac{4}{3}}([0,T],L^2_\sharp)$.

Proof of Lemma 1. The sequence $(\overline{h}_\delta)_{\delta 0}$ is obviously bounded in the space $L^{\frac{4}{3}}([0,T],L^2_\sharp)$. Thus from this sequence one can extract a subsequence (still denoted by $(\overline{h}_\delta)_{\delta>0}$) which weakly converges to some $g \in L^{\frac{4}{3}}([0,T],L^2_\sharp)$. By taking $\phi \in C^\infty([0,T]\times Q)$, space periodic, one obtains after two integrations by parts in space in the equation satisfied by $\overline{h}_\delta$, $A\overline{h}_\delta = h_\delta$,

$$-\delta\int_0^T\int_Q \Delta\phi\,\overline{h}_\delta + \int_0^T\int_Q \phi\, h_\delta = \int_0^T\int_Q \phi\, h_\delta. \tag{35}$$

In (35), $\displaystyle\int_0^T\int_Q \Delta\phi\,\overline{h}_\delta$ remains bounded because of the bound of $(\overline{h}_\delta)_{\delta>0}$ in the space $L^{\frac{4}{3}}([0,T],L^2_\sharp)$. Thus, one has

$$\lim_{\delta\to 0}\delta\int_0^T\int \Delta\phi\,\overline{h}_\delta = 0.$$

Passing to the limit in (35) for $\delta\to 0$ yields

$$\int_0^T\int \phi\, g = \int_0^T\int \phi\, h, \tag{36}$$

an equality which holds for each ϕ space periodic and smooth. Therefore $g = h$. The possible weak limit being unique, all the sequence converges. □

Lemma 2. *The sequence* $(q_\delta)_{\delta>0}$ *is bounded in the space* $L^{\frac{4}{3}}([0,T],L^2_\sharp)$.

Proof of Lemma 2 . Taking the divergence of equation (6) yields

$$-\Delta q_\delta = \nabla\cdot(\nabla\cdot\overline{\mathbf{w}_\delta\mathbf{w}_\delta}) - \nabla\cdot\mathbf{f}, \tag{37}$$

with periodic conditions and mean value equal to zero, $\displaystyle\int_Q q_\delta = 0$.

The energy estimate for $\mathbf{w}$ shows that the sequence $(\mathbf{w}_\delta)_{\delta>0}$ is bounded in the space $L^\infty([0,T],V_2)\cap L^2([0,T],V_1)$ included in $L^\infty([0,T],(L^2_\sharp)^3)\cap L^2([0,T],(L^6_\sharp)^3)$. Hölder inequality implies

$$L^\infty([0,T],(L^2_\sharp)^3)\cap L^2([0,T],(L^6_\sharp)^3)\subset L^{\frac{8}{3}}([0,T],(L^4_\sharp)^3).$$

Consequently, the sequence $(\mathbf{w}_\delta\mathbf{w}_\delta)_{\delta>0}$ is bounded in $L^{\frac{4}{3}}([0,T],(L^2_\sharp)^9)$, from which one deduces that

$$\text{The sequence } (\overline{\mathbf{w}_\delta\mathbf{w}_\delta})_{\delta>0} \text{ is bounded in } L^{\frac{4}{3}}([0,T],(L^2_\sharp)^9). \tag{38}$$

One concludes from the classical elliptic theory and from (37) that $(q_\delta)_{\delta>0}$ is bounded in

$$L^{\frac{4}{3}}([0,T],L^2_\sharp) + L^2([0,T],L^2_\sharp)\subset L^{\frac{4}{3}}([0,T],L^2_\sharp)$$

and the lemma is proven. □

Remark 2. Note that it is easy deduced from the considerations above that

$$\text{the sequence } (\partial_t\mathbf{w}_\delta)_{\delta>0} \text{ is bounded in } L^{\frac{4}{3}}([0,T],W_1'), \tag{39}$$

where we recall that the space W_1 is defined by (8).

Proof of Theorem 4. Because of the bound of the sequence $(\mathbf{w}_\delta)_{\delta>0}$ in the space $L^2([0,T],V_1)$, uniform in δ, one can extract a subsequence (still denoted $(\mathbf{w}_\delta)_{\delta>0}$) which converges weakly to some $\mathbf{u}\in L^2([0,T],V_1)$ when δ goes to zero. Thanks to Lemma 2, one can extract from the sequence $(q_\delta)_{\delta>0}$ a subsequence (still denoted by $(q_\delta)_{\delta>0}$) which converges weakly to some p in $L^{\frac{4}{3}}([0,T],L^2_\sharp)$.

We shall show that $(\mathbf{u},p)$ is a weak solution to the Navier-Stokes equations in $I\!R^3$ with periodic boundary conditions. We shall pass to the limit in each term of the equations for proving our claim. Let $(\mathbf{v},q)\in C^\infty([0,T],\mathcal{W})\times C^\infty([0,T],C^\infty_\sharp)$. One has

$$\left\{\begin{array}{l}\displaystyle\int_0^T\int_Q \partial_t\mathbf{w}_\delta.\mathbf{v}-\int_0^T\int_Q\overline{\mathbf{w}_\delta\mathbf{w}_\delta}\,.\nabla\mathbf{v}+\nu\int_0^T\int_Q\nabla\mathbf{w}_\delta.\nabla\mathbf{v}\\ \displaystyle-\int_0^T\int_Q q_\delta\,\nabla\cdot\mathbf{v}=\int_0^T<\mathbf{f},\mathbf{v}>,\\ \displaystyle\int_0^T\int_Q\nabla q.\mathbf{w}_\delta=0.\end{array}\right. \tag{40}$$

Note at first that by an integration by parts one has

$$0=\int_0^T\int_Q q\,\nabla\cdot\mathbf{w}_\delta=-\int_0^T\int_Q\nabla q.\mathbf{w}_\delta\to-\int_0^T\int_Q\nabla q.\mathbf{u}=0. \tag{41}$$

Thus

$$\nabla\cdot\mathbf{u}=0. \tag{42}$$

The weak convergence of the sequence $(q_\delta)_{\delta>0}$ in $L^{\frac{4}{3}}([0,T],L^2_\sharp)$ yields

$$\lim_{\delta\to0}\int_0^T\int_Q q_\delta\,\nabla\cdot\mathbf{v}=\int_0^T\int_Q p\,\nabla\cdot\mathbf{v}. \tag{43}$$

Estimate (39) makes sure that one can extract a subsequence from the sequence $(\partial_t\mathbf{w}_\delta)_{\delta>0}$ which converges weakly in $L^{\frac{4}{3}}([0,T],W_1')$ to some $\mathbf{g}$. When $\mathbf{v}$ has a compact support in time,

$$\int_0^T\int_Q\partial_t\mathbf{w}_\delta.\mathbf{v}=-\int_0^T\int_Q\mathbf{w}_\delta.\partial_t\mathbf{v}.$$

Consequently,

$$\lim_{\delta\to0}\int_0^T\int_Q\mathbf{w}_\delta.\partial_t\mathbf{v}=\int_0^T\int_Q\mathbf{u}.\partial_t\mathbf{v}=-<\partial_t\mathbf{u},\mathbf{v}>=-<\mathbf{g},\mathbf{v}>,$$

the last braket having to be considered in the sense of the distributions. Then $\mathbf{g}=\partial_t\mathbf{u}$ in the sense of the distributions. Then $\partial_t\mathbf{u}\in L^{\frac{4}{3}}([0,T],W_1')$ and one has

$$\lim_{\delta\to0}\int_0^T\int_Q\partial_t\mathbf{w}_\delta.\mathbf{v}=\int_0^T\int_Q\partial_t\mathbf{u}\,.\mathbf{v}. \tag{44}$$

Finally one has obviously,

$$\lim_{\delta\to0}\int_0^T\int_Q\nabla\mathbf{w}_\delta.\nabla\mathbf{v}=\int_0^T\int_Q t\nabla\mathbf{u}\,.\nabla\mathbf{v}. \tag{45}$$

It remains to pass to the limit in the nonlinear term $\nabla\cdot\overline{\mathbf{w}_\delta\mathbf{w}_\delta}$.

We already know that the sequence $(\mathbf{w}_\delta\mathbf{w}_\delta)_{\delta>0}$ is bounded in $L^{\frac{4}{3}}([0,T],(L^2_\sharp)^9)$ (see the proof of lemma 2). Thus, up to a subsequence, it converges weakly to some

Ψ in $L^{\frac{4}{3}}([0,T],(L^2_\sharp)^9)$. Applying lemma 1, $(\overline{\mathbf{w}_\delta\mathbf{w}_\delta})_{\delta>0}$ converges weakly to Ψ in $L^{\frac{4}{3}}([0,T],(L^2_\sharp)^9)$. We have to prove that $\Psi = \mathbf{uu}$.

The bound of the sequence $(\mathbf{w}_\delta)_{\delta>0}$ in $L^2([0,T],V_1)$ combined with the bound of $(\partial_t\mathbf{w}_\delta)_{\delta>0}$ in $L^{\frac{4}{3}}([0,T],W_1')$ make sure that the sequence $(\mathbf{w}_\delta)_{\delta>0}$ is compact in $L^{\frac{4}{3}}([0,T],(L^2_\sharp)^3)$, and the convergence is strong in $L^{\frac{4}{3}}([0,T],(L^2)^3)$ by using Aubin-Lions's Lemma (one point that we claimed in the statement of Theorem 4). By classical arguments using inverse Lebesgue's Theorem, see e.g., [14], one can extract a subsequence (without change of the notation) which converges *a.e.* in $[0,T]\times Q$ to $\mathbf{w}$. Consequently, $(\mathbf{w}_\delta\mathbf{w}_\delta)_{\delta>0}$ converges *a.e* to $\mathbf{uu}$ and this suffices to make sure that $\Psi = \mathbf{uu}$. Then

$$\lim_{\delta\to 0}\int_0^T\int_Q \overline{\mathbf{w}_\delta\mathbf{w}_\delta}\,.\,\nabla\mathbf{v} = \int_0^T\int_Q \mathbf{uu}\,.\,\nabla\mathbf{v}. \tag{46}$$

When one puts together (40), (41), (43), (44), (45) and (46), one has

$$\begin{cases} \displaystyle\int_0^T\int_Q \partial_t\mathbf{u}.\mathbf{v} - \int_0^T\int_Q \mathbf{uu}\,.\nabla\mathbf{v} + \nu\int_0^T\int_Q \nabla\mathbf{u}.\nabla\mathbf{v} \\ \displaystyle -\int_0^T\int_Q p\,\nabla\cdot\mathbf{v} = \int_0^T <\mathbf{f},\mathbf{v}>, \\ \displaystyle\int_0^T\int_Q \nabla q.\mathbf{u} = 0. \end{cases} \tag{47}$$

Therefore $(\mathbf{u},p)$ is a weak solution to the Navier-Stokes equations in $I\!R^3$ with periodic boundary conditions. The proof of Theorem 4 is complete. □

3.3. Verifiability of the Model.

Theorem 5. *Suppose the true solution of the NSE satisfies the regularity condition* $||\nabla\mathbf{u}|| \in L^4(0,T)$*. Let* $\boldsymbol{\tau} := \overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{uu}$*. Then* $\overline{\mathbf{u}} - \mathbf{w}$ *satisfies*

$$\begin{aligned} &||(\overline{\mathbf{u}}-\mathbf{w})(t,\cdot))||^2 + \nu\int_0^t ||\nabla(\overline{\mathbf{u}}-\mathbf{w})(s,\cdot)||^2 ds \\ &\leq C\nu^{-1}e^{\nu^{-3}A(t)}\int_0^t ||\boldsymbol{\tau}||^2 ds, \end{aligned} \tag{48}$$

where $A(t) := \displaystyle\int_0^t ||\nabla u||^4 ds$.

In the result above, $||\cdot||$ stand for the L^2 spacial norm.

Remark 3. It is straightforward to weaken the assumption $||\nabla\mathbf{u}|| \in L^4(0,T)$ to the Serrin [23] condition that $\mathbf{u} \in L^r(0,T;(L^5_\sharp)^3)$. The main problems with the estimate (48), however, are (1) the multiplier $e^{-\nu^{-3}A(t)}$ is huge for small ν, and (2) the natural analytic norm for measuring the consistency error is $L^2([0,T],(L^2_\sharp)^9)$. However, (as we shall see) giving analytic bounds on $||\boldsymbol{\tau}||_{L^2([0,T],(L^2_\sharp)^9)}$ depends (since $\boldsymbol{\tau}$ is quadratic in $\mathbf{u}$) on á priori bounds on first and second derivatives of $\mathbf{u}$.

3.4. **Accuracy of the Model. Proof of Theorem 5.** As noted in the introduction $\mathbf{e} = \overline{\mathbf{u}} - \mathbf{w}$ satisfies (in the sense of its variational formulation), $\nabla \cdot \mathbf{e} = 0, \mathbf{e}(0) = 0$ and

$$\mathbf{e}_t + \nabla \cdot (\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{w}\mathbf{w}}) + \nabla(\overline{p} - q) - \nu \Delta \mathbf{e} = \nabla \cdot \overline{\boldsymbol{\tau}},$$

where $\boldsymbol{\tau} := \overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{u}\mathbf{u}$. This is exactly the perturbation equation for the model we study. Under the stated assumption on $\mathbf{u}$, $\mathbf{u}$ is a strong solution of the NSE and $\mathbf{w}$ is a strong solution of the model. Thus, this equation holds in the strong sense. Taking the inner product with $A\mathbf{e}$ and following the proof of the model's basic energy estimate gives

$$\frac{1}{2}\frac{d}{dt}\{||\mathbf{e}||^2 + \delta^2||\nabla \mathbf{e}||^2\} + \nu\{||\nabla \mathbf{e}||^2 + \delta^2||\Delta \mathbf{e}||^2\} + (\nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{w}\mathbf{w}), \mathbf{e}) \leq -(\boldsymbol{\tau}, \nabla \mathbf{e})$$

Writing $\overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{w}\mathbf{w} = \mathbf{w}\mathbf{e} + \mathbf{e}\overline{\mathbf{u}}$ and using $(\nabla \cdot \mathbf{w}\mathbf{e}, \mathbf{e}) = 0$ and $|(\boldsymbol{\tau}, \nabla \mathbf{e})| \leq \frac{\nu}{2}||\nabla \mathbf{e}||^2 + \frac{1}{2\nu}||\boldsymbol{\tau}||^2$ we then have

$$\begin{aligned}\frac{d}{dt}\{||\mathbf{e}||^2 + \delta^2||\nabla \mathbf{e}||^2\} + \nu\{||\nabla \mathbf{e}||^2 + \delta^2||\Delta \mathbf{e}||^2\} &\leq \nu^{-1}||\boldsymbol{\tau}||^2 - (\mathbf{e} \cdot \nabla \overline{\mathbf{u}}, \mathbf{e}) \\ &\leq \nu^1||\boldsymbol{\tau}||^2 + C||\nabla \mathbf{e}||^{3/2}||\mathbf{e}||^{1/2}||\nabla \overline{\mathbf{u}}||.\end{aligned}$$

Thus, using $ab \leq \epsilon a^{4/3} + C\epsilon^{-3}b^4$, we obtain

$$\frac{d}{dt}\{||\mathbf{e}||^2 + \delta^2||\nabla \mathbf{e}||^2\} + \nu\{||\nabla \mathbf{e}||^2 + \delta^2||\Delta \mathbf{e}||^2\} \leq C\nu^{-1}||\boldsymbol{\tau}||^2 + C\nu^{-3}||\nabla \overline{\mathbf{u}}||^4||\mathbf{e}||^2.$$

Gronwall's inequality then implies

$$\begin{aligned}&||\mathbf{e}(t,\cdot)||^2 + \delta^2||\nabla \mathbf{e}(t,\cdot)||^2 + \nu \int_0^t ||\nabla \mathbf{e}(s,\cdot)||^2 + \delta^2||\Delta \mathbf{e}(s,\cdot)||^2 ds \\ \leq \quad & C\nu^{-1}e^{\nu^{-3}A(t)} \int_0^t ||\boldsymbol{\tau}(s,\cdot)||^2 ds,\end{aligned}$$

where $A(t) = \displaystyle\int_0^t ||\nabla \mathbf{u}(s,\cdot)||^4 \, ds$. Since we are searching for a result which is uniform in δ, the δ-terms on the inequalities LHS are dropped and, on the RHS, the stability bound

$$||\nabla \overline{\mathbf{u}}|| \leq ||\nabla \mathbf{u}||,$$

(see lemma 3 and estimate (50) below) is used with the assumption that $||\nabla \mathbf{u}|| \in L^4(0,T)$. □

4. **Consistency error estimates.** Recall that

$$\boldsymbol{\tau}(\mathbf{u}, \mathbf{u}) := \overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{u}\mathbf{u}. \tag{49}$$

In this section, we shall give bounds on $||\boldsymbol{\tau}||_{L^1([0,T]\times Q)}$ as $\delta \to 0$ in the general case. An estimate of $||\boldsymbol{\tau}||_{L^2((0,T)\times Q)}$ will be provided under additional regularity properties.

First, classical results on singular perturbations are needed. They are recalled in the next subsection.

4.1. **Some singular perturbations results.**

Lemma 3. *Let* $\varphi \in L^p_\sharp$, $1 \le p < \infty$. *Then*

$$||A_\delta^{-1}\varphi||_{L^p_\sharp} \le ||\varphi||_{L^p_\sharp}. \tag{50}$$

Moreover, when $p > 1$, $(A_\delta^{-1}\varphi)_{\delta>0}$ *converges towards* φ *strongly in* $L^p_\sharp$.

This is a direct consequence of classical stability results in elliptic theory. For completeness, we give a short proof below, condensed from [13].

Proof. Put $\overline{\varphi} = A_\delta^{-1}\varphi$. Recall that

$$-\delta^2\Delta\overline{\varphi} + \overline{\varphi} = \varphi. \tag{51}$$

Take $\psi(\overline{\varphi}) = \overline{\varphi}|\overline{\varphi}|^{p-2}$ as test function in (4.3) when $p > 1$, (with the modification that if $p = 1$ we take $\psi(\overline{\varphi}) = \text{sgn}(\overline{\varphi})$ and use some technical tools) and integrate by parts. This yields

$$\delta^2 \int_Q \psi'(\overline{\varphi})|\overline{\varphi}|^2 + \int_Q |\overline{\varphi}|^p = \int_Q \varphi\psi(\overline{\varphi}). \tag{52}$$

Because ψ is a non decreasing function, we can deduce from (52) that

$$\int_Q |\overline{\varphi}|^p \le \int_Q \varphi\psi(\overline{\varphi}). \tag{53}$$

Inequality (50) is directly deduced from (53) when $p = 1$. Assume now that $p > 1$. Then (53) yields

$$\int_Q |\overline{\varphi}|^p \le \int_Q |\varphi|\,|\overline{\varphi}|^{p-1}. \tag{54}$$

We use Hölder inequality in the r.h.s of (54). Then (54) becomes

$$||\overline{\varphi}||^p_{L^p_\sharp} \le ||\varphi||_{L^p_\sharp}||\overline{\varphi}||^{p-1}_{L^p_\sharp}. \tag{55}$$

yielding (50). □

The next result is easily proven as well, see e.g., [15].

Lemma 4. *Let* $\varphi \in L^2_\sharp$. *Then*

$$||\overline{\varphi} - \varphi||_{L^2_\sharp} \le \frac{\delta}{\sqrt{2}}||\nabla\varphi||_{L^2_\sharp}. \tag{56}$$

4.2. L^1 **Consistency error.** Throughout this subsection, one assumes that $\mathbf{f} \in L^2([0,T], V_1')$. Let $\mathbf{u}$ be any solution to the Navier-Stokes equation. Recall that

$$\boldsymbol{\tau}(\mathbf{u},\mathbf{u}) := \overline{\mathbf{u}}\,\overline{\mathbf{u}} - \mathbf{u}\mathbf{u}. \tag{57}$$

Introduce

$$W(T) = \int_Q |\mathbf{u}_0(x)|^2 dx + \frac{1}{\nu}\int_0^T ||\mathbf{f}||_{V_1'}dt. \tag{58}$$

Lemma 5. *The following holds*

$$||\boldsymbol{\tau}||_{L^1([0,T],(L^1_\sharp)^9)} \le \sqrt{2}\,\delta\, T^{\frac{1}{2}}\nu^{-\frac{1}{2}}W(T). \tag{59}$$

Proof. Because $\mathbf{u}$ is a solution to the Navier-Stokes equation then the classical energy estimate holds for all $t \le T$,

$$\int_Q |\mathbf{u}(t,x)|^2 dx + \nu \int_0^t \int_Q |\nabla \mathbf{u}(t',x)|^2 dx dt' \le W(T). \tag{60}$$

Next write

$$\boldsymbol{\tau} = \overline{\mathbf{u}}(\overline{\mathbf{u}} - \mathbf{u}) + \mathbf{u}(\overline{\mathbf{u}} - \mathbf{u}).$$

Thus, by using (50), one has

$$||\boldsymbol{\tau}||_{L^1([0,T],(L^1_\sharp)^9)} \le 2||\mathbf{u}||_{L^2([0,T],(L^2_\sharp)^3)}||\overline{\mathbf{u}} - \mathbf{u}||_{L^2([0,T],(L^2_\sharp)^3)}.$$

Therefore, (59) is a consequence of (56) combined to (60). □

4.3. L^2 **Consistency error.** Because estimate (3.16) involves L^2 norm of $\boldsymbol{\tau}$, we are lead to seek for an estimate of this quantity. As already mentioned in Theorem 5, a regularity assumption on the velocity is needed to derive estimate (48). Due to the nature of the filter, an other regularity assumption has to be introduced. This regularity is known in the 2D case, but not in the 3D case. We stay here in the 3D case.

However, we stress that such kind of estimates can be found in [21] and references therein, in the 1D case and for C^∞ fields. Our result complements these since it considers solutions with the (limited) regularity typical of solutions of the NSE.

Proposition 1. *Let* $\mathbf{u}$ *be a solution to the NSE. Assume that*

$$\mathbf{u} \in L^4([0,T],(L^4_\sharp)^3) \cap L^1([0,T],(H^2_\sharp)^3). \tag{61}$$

Then one has

$$||\boldsymbol{\tau}||_{L^2([0,T],(L^2_\sharp)^9)} \le C\delta, \tag{62}$$

where $C = C(||\mathbf{u}||_{L^4([0,T],(L^4_\sharp)^3)}, ||\mathbf{u}||_{L^1([0,T],(H^2_\sharp)^3)})$.

Proof. Observe first that by Cauchy-Schwarz inequality and (50),

$$||\boldsymbol{\tau}||_{L^2([0,T],(L^2_\sharp)^9)} \le 2||\mathbf{u}||_{(L^4_\sharp)^3}||\overline{\mathbf{u}} - \mathbf{u}||_{(L^4_\sharp)^3}. \tag{63}$$

Next, it is known that

$$||\overline{\mathbf{u}} - \mathbf{u}||_{(L^4_\sharp)^3} \le C||\overline{\mathbf{u}} - \mathbf{u}||^{\frac{1}{4}}_{(L^2_\sharp)^3}||\nabla(\overline{\mathbf{u}} - \mathbf{u})||^{\frac{3}{4}}_{(L^2_\sharp)^9}. \tag{64}$$

Because we are in a periodic case, then (56) applies to $||\nabla(\overline{\mathbf{u}} - \mathbf{u})||^{\frac{3}{4}}_{(L^2_\sharp)^9}$. Then (62) is deduced from (63) combined to (64) and (56), a time integration and the use of Hölder inequality.

5. **Vortex Structures of the Model.** In under-resolved simulation of fluid flow at higher Reynolds numbers vortices often appear that seem to be spurious. Are these the result of backscatter in the true flow equations, so that the extra vortices are physically correct for a small physical perturbation? Are they non-physical vortices excited by truncation error terms? Are they non-physical vortices that appear as solutions of a turbulence model used that do not reflect appropriate averages of the true flow's eddies? Studies of the second question have been pioneered by Brown and Minon [2] and Drikakis and Smolarkiewicz [4]. Our aim here is to begin considering the third question in an (admittedly quite simplified) setting which admits analytical attack.

Taking the curl of the model we obtain an equation for the vorticity predicted by the model. If the model is appropriate, $\nabla \times \mathbf{w}$ should be a non-spurious approximation of the true vorticity $\nabla \times \mathbf{u}$. The question is: does the model so-derived for $\mathbf{w}$ make non-physical predictions of $\nabla \times \mathbf{u}$? Since such questions center on the relationship between $\nabla \times \mathbf{u}$ and $\nabla \times \mathbf{w}$ we first note that equations for the vorticity predicted by the model are easily derived. Indeed, taking the curl of the LES model shows that $\nabla \times \mathbf{w} =: \boldsymbol{\omega}$ satisfies, $\boldsymbol{\omega}(0) = \overline{\nabla \times \mathbf{u}_0}$ and

$$\boldsymbol{\omega}_t + \overline{\mathbf{w} \cdot \nabla \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \mathbf{w}} - \nu \Delta \boldsymbol{\omega} = \overline{\nabla \times \mathbf{f}}, \text{ if } \boldsymbol{\omega} \subset \mathbf{R}^3, \tag{65}$$

and

$$\boldsymbol{\omega}_t + \overline{\mathbf{w} \cdot \nabla \boldsymbol{\omega}} - \nu \Delta \boldsymbol{\omega} = \overline{\nabla \times \mathbf{f}}, \text{ if } \boldsymbol{\omega} \subset \mathbf{R}^2. \tag{66}$$

First, we consider the simplest (and easiest) case: we show that if $\nabla \times \mathbf{u}_0 = \nabla \times \mathbf{f} \equiv 0$ then $\boldsymbol{\omega} \equiv 0$, i.e., no spurious vorticity is generated by the model of the nonlinear interaction.

Proposition 2. *Let* $\mathbf{f}$ *and* $\mathbf{u}_0$ *be smooth. If* $\nabla \times \mathbf{u}_0 \equiv 0, \nabla \times \mathbf{f} \equiv 0$ *then* $\boldsymbol{\omega} = \nabla \times \mathbf{w} \equiv 0$.

Proof: Recall first that the symbol $||\cdot||$ without subscript denote general L^2 spacial norms. First we note that by the estimates of the previous sections, $\mathbf{w}$ (and hence $\boldsymbol{\omega}$) is smooth. Adapting the proof of the energy estimate of the model, take the inner product of (65) with $A\boldsymbol{\omega}$. This gives, after integrations by parts,

$$\begin{aligned} & \frac{1}{2}\ \frac{d}{dt}\{||\boldsymbol{\omega}||^2 + \delta^2||\nabla\boldsymbol{\omega}||^2\} + \nu\{||\nabla\boldsymbol{\omega}||^2 + \delta^2||\Delta\boldsymbol{\omega}||^2\} \\ = \ & -(\overline{\mathbf{w} \cdot \nabla \boldsymbol{\omega}}, A\boldsymbol{\omega}) - (\overline{\boldsymbol{\omega} \cdot \nabla \mathbf{w}}, A\boldsymbol{\omega}) \\ = \ & -(\boldsymbol{\omega} \cdot \nabla \mathbf{w}, \boldsymbol{\omega}) \leq ||\nabla \mathbf{w}||\ ||\nabla \boldsymbol{\omega}||^2. \end{aligned}$$

Since $||\nabla \mathbf{w}(t, \cdot)|| \in L^\infty(0, T)$, the result follows by Gronwall's inequality. ☐

Naturally, it is more important that small perturbations remain small. This fact also follows by essentially the same energy argument.

Proposition 3. *Let* $\mathbf{u}_0$ *and* $\mathbf{f}$ *be smooth and suppose*

$$||\nabla \times \mathbf{u}_0|| \leq \epsilon, \quad ||\nabla \times \mathbf{f}||_{L^\infty([0,T],W_2)} \leq \epsilon.$$

Then $||\boldsymbol{\omega}(t, \cdot)||$ *satisfies in* $3d$:

$$\frac{1}{2}\{||\boldsymbol{\omega}(t)||^2 + \delta^2||\nabla\boldsymbol{\omega}(t)||^2\} + \int_0^t \nu||\nabla\boldsymbol{\omega}||^2 + \delta^2\nu||\Delta\boldsymbol{\omega}||^2 ds \leq \left[\left(1 + \frac{t}{2}\right) e^{A(t)}\right]\epsilon^2,$$

where $A(t) = \dfrac{t}{2} + \displaystyle\int_0^t ||\nabla \mathbf{w}||(s) ds < \infty$*, and, in* $2d$:

$$\frac{1}{2}\{||\boldsymbol{\omega}(t)||^2 + \delta^2||\nabla\boldsymbol{\omega}(t)||^2\} + \int_0^t \nu||\nabla\boldsymbol{\omega}||^2 + \delta^2\nu||\Delta\boldsymbol{\omega}||^2 ds \leq (1 + t)\epsilon^2.$$

Proof: First we note that, by integration by parts,

$$\begin{aligned} & ||\boldsymbol{\omega}(0)||^2 + \delta^2||\nabla\boldsymbol{\omega}(0)||^2 = (\boldsymbol{\omega}(0), \boldsymbol{\omega}(0) - \delta^2\Delta\boldsymbol{\omega}(0)) \\ = \ & (\boldsymbol{\omega}(0), A\boldsymbol{\omega}(0)) = (\overline{\nabla \times \mathbf{u}_0}, A\ \overline{\nabla \times \mathbf{u}_0}) \\ = \ & (\overline{\nabla \times \mathbf{u}_0}, \nabla \times \mathbf{u}_0) \leq ||\nabla \times \mathbf{u}_0||^2. \end{aligned}$$

By the same argument as in Proposition 1 we have, in 3d,

$$\begin{aligned} & \frac{1}{2}\ \frac{d}{dt}\{||\boldsymbol{\omega}||^2+\delta^2||\nabla\boldsymbol{\omega}||^2\}+\nu\{||\nabla\boldsymbol{\omega}||^2+\delta^2||\Delta\boldsymbol{\omega}||^2\} \\ = \quad & -(\boldsymbol{\omega}\cdot\nabla\mathbf{w},\boldsymbol{\omega})+(\nabla\times\mathbf{f},\boldsymbol{\omega}) \\ \leq \quad & \left(\frac{1}{2}+||\nabla\mathbf{w}||\right)||\boldsymbol{\omega}||^2+\frac{1}{2}||\nabla\times\mathbf{f}||^2 \end{aligned}$$

Thus, by Gronwall's inequality

$$\begin{aligned} & \frac{1}{2}\{||\boldsymbol{\omega}(t)||^2+\delta^2||\nabla\boldsymbol{\omega}(t)||^2\}+\int_0^t\nu\{||\nabla\boldsymbol{\omega}||^2+\delta^2||\Delta\boldsymbol{\omega}||^2\}ds \\ \leq \quad & e^{A(t)}||\nabla\times\mathbf{u}_0||^2+\frac{1}{2}e^{A(t)}\int_0^t||\nabla\times\mathbf{f}||^2ds \\ \leq \quad & e^{A(t)}\left(1+\frac{1}{2}t\right)\epsilon^2, \end{aligned}$$

where $A(t)=\dfrac{1}{2}+\displaystyle\int_0^t||\nabla\mathbf{w}||(s)ds \quad (<\infty)$.

In 2-d, the same argument can be used but the term $(\boldsymbol{\omega}\cdot\nabla\mathbf{w},\boldsymbol{\omega})$ does not appear on the RHS. Thus, the 2-d estimate does not display exponential growth. □

Remark: An Open Problem. There is one more case for which the question can be formulated mathematically. In 2-d, the true vorticity equation is:

$$(\nabla\times\mathbf{u})_t+\mathbf{u}\cdot\nabla(\nabla\times\mathbf{u})-\nu\Delta(\nabla\times\mathbf{u})=\nabla\times\mathbf{f}.$$

This equation satisfies a maximum principle. Thus, when $\nabla\times\mathbf{u}_0$ and $\nabla\times\mathbf{f}$ have one sign, $\nabla\times\mathbf{u}$ must also have one sign. Thus, it is reasonable to ask the question: if the initial condition and the body force exert only a counterclockwise rotation force on the flow, does the LES model (correctly) predict that only a counterclockwise rotation of the flow results?

The above proofs can be attempted to be combined with generalized maximum principle arguments. Unfortunately, the obvious combination fails (for a subtle reason we describe below). The mathematical treatment of this last question is an open problem.

To understand the point of failure, define $\boldsymbol{\omega}_-:=-\max\{-\boldsymbol{\omega},0\}$. Taking the inner product of (66) with $A\boldsymbol{\omega}_-$ (and ignoring boundary terms) gives (following the above proofs)

$$\frac{1}{2}\frac{d}{dt}\{||\boldsymbol{\omega}_-||^2+\delta^2||\nabla\boldsymbol{\omega}_-||^2\}+\nu||\nabla\boldsymbol{\omega}||^2+\nu\delta^2(\Delta\boldsymbol{\omega},\Delta\boldsymbol{\omega}_-)=(\nabla\times\mathbf{f},\boldsymbol{\omega}_-)\leq 0.$$

Unfortunately, $(\Delta\boldsymbol{\omega},\Delta\boldsymbol{\omega}_-)$ does not have one sign. A calculation of $(\Delta\boldsymbol{\omega},\Delta\boldsymbol{\omega}_-)$ from the definition of $\Delta\boldsymbol{\omega}_-$ as a distribution in $H^{-1}(\boldsymbol{\omega})$ gives

$$(\Delta\boldsymbol{\omega},\Delta\boldsymbol{\omega}_-)=\int_{\boldsymbol{\omega}}|\Delta\boldsymbol{\omega}_-|^2dx-\int_{\partial\mathrm{supp}(\boldsymbol{\omega}_-)}(\Delta\boldsymbol{\omega})(s)\nabla\boldsymbol{\omega}_-\cdot\hat{n}(s)ds.$$

The last term on the RHS can have any sign. Thus, direct proof fails.

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